Computing with Daubechies' Wavelets

A.B. Olde Daalhuis
CWI, P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

In the first two sections of this paper we describe a construction of the compactly supported Daubechies' wavelets. In the third section we show how a Discrete Wavelet Transform of a function can be obtained. We end this paper with some interesting numerical illustrations.

1. Wavelets with finitely many non-zero filtercoefficients

In the theory of discrete wavelets the equation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \phi(2x - n) \tag{1.1}$$

plays a fundamental role. With the farther wavelet ϕ, satisfying this equation, we have the accompanying mother wavelet ψ defined by

$$\psi(x) := \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^n h_{1-n} \phi(2x - n). \tag{1.2}$$

In Daubechies [1] for the first time a construction is given of wavelets resulting in compactly supported orthonormal wavelet bases. See also Heijmans [this volume].

It is not difficult to put conditions on h_n of (1.1) in order to obtain compactly supported functions ϕ and ψ. We only need that just a finite number of $\{h_n\}$ are different from zero. In this section we give a proof of this property.

We use the fact that the solution ϕ of equation (1.1) can be constructed by the following iteration scheme. Let

$$\eta_0(x) = \begin{cases} 1 & \text{as } -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{elsewhere.} \end{cases} \tag{1.3}$$

That is $\eta_0(x)$ is the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$. Next we define a sequence of functions η_l, $l = 1, 2, \ldots$, by writing

$$\eta_l(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \eta_{l-1}(2x - n). \tag{1.4}$$

Then we have

$$\phi(x) = \lim_{l \to \infty} \eta_l(x). \tag{1.5}$$

For a proof of this constructive procedure, which may also be used to draw pictures of solutions of (1.1), see [1].

Now, let just a finite number of filter coefficients h_n of (1.1) be non-zero. That is, assume that we have two integer numbers N_-, N_+ and that

$$h_n = 0 \qquad \begin{matrix} n < N_-, \\ n > N_+. \end{matrix} \tag{1.6}$$

It is easily verified that the functions η_l defined by (1.3) and (1.4) have compact support. We have in fact

$$\mathrm{supp}(\eta_l) = [N_{l,-}, N_{l,+}],$$

with

$$\begin{aligned} N_{0,-} &= -\tfrac{1}{2}, & N_{0,+} &= \tfrac{1}{2}, \\ N_{l,-} &= \tfrac{1}{2}(N_{l-1,-} + N_-), & N_{l,+} &= \tfrac{1}{2}(N_{l-1,+} + N_+). \end{aligned}$$

Thus we have

$$\begin{matrix} N_{l,-} \to N_- \\ N_{l,+} \to N_+ \end{matrix} \qquad \text{as } l \to \infty,$$

and it follows that

$$\mathrm{supp}(\phi) \subset [N_-, N_+], \tag{1.7}$$

and with (1.2)

$$\mathrm{supp}(\psi) \subset [\tfrac{1}{2}(1 - N_+ + N_-), \tfrac{1}{2}(1 + N_+ - N_-)]. \tag{1.8}$$

In a recent paper of LEMARIÉ-RIEUSSET & MALGOUYRES [2] it is proven that the support of ϕ is indeed a connected interval.

2. DAUBECHIES' CONSTRUCTION OF COMPACTLY SUPPORTED WAVELETS

In this section we give a few theorems which are relevant in the theory of compactly supported wavelets. These theorems are proven in [1].

THEOREM 1. (Daubechies) *Let h_n be a sequence such that*
(i) $\sum_n |h_n||n|^\varepsilon < \infty$ *for some* $\varepsilon > 0$,
(ii) $\sum_n h_{n-2k} h_{n-2l} = \delta_{kl}$,
(iii) $\sum_n h_n = \sqrt{2}$.
Suppose also that $H(\xi) = 2^{-\frac{1}{2}} \sum_n h_n e^{-in\xi}$ *can be written as*

$$H(\xi) = [\tfrac{1}{2}(1 + e^{-i\xi})]^N \big[\sum_n f_n e^{-in\xi}\big], \tag{2.1}$$

where
(iv) $\sum_n |f_n||n|^\varepsilon < \infty$ *for some* $\varepsilon > 0$,
(v) $\sup_{\xi \in \mathbb{R}} |\sum_n f_n e^{-in\xi}| < 2^{N-1}$.

Define

$$\begin{aligned} g_n &= (-1)^n h_{1-n}, \\ \widehat{\phi}(\xi) &= \textstyle\prod_{j=1}^{\infty} H(2^{-j}\xi), \\ \psi(x) &= \sqrt{2}\textstyle\sum_n g_n \phi(2x-n). \end{aligned}$$

Then the $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k)$ *define a multiresolution analysis, and the* $\{\psi_{jk}\}$ *are the associated orthonormal wavelet basis.*

The function $\widehat{\phi}$ denotes the Fourier transform of the function ϕ:

$$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi}\phi(x)dx.$$

The function $H(\xi)$ introduced in this theorem is of fundamental importance in the theory. Observe that $H(\xi)$ generates the function $\widehat{\phi}(\xi)$. So, when H is known, $\widehat{\phi}$ and, hence, ϕ can be constructed.

Also, the number N in (2.1) is crucial: when N is large the wavelet has interesting properties with respect to approximations with the mother wavelets $\{\psi_{jk}\}$.

REMARK. Starting with finitely many h_n, we obtain finitely many f_n, and *(i)* and *(iv)* of Theorem 1 are obviously fulfilled.

EXAMPLES:

(1) $$h_0 = h_1 = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad H(\xi) = \tfrac{1}{2}(1 + e^{-i\xi})$$

(2) $$\begin{aligned} &h_0 = \tfrac{1+\sqrt{3}}{4\sqrt{2}},\ h_1 = \tfrac{3+\sqrt{3}}{4\sqrt{2}},\ h_2 = \tfrac{3-\sqrt{3}}{4\sqrt{2}},\ h_3 = \tfrac{1-\sqrt{3}}{4\sqrt{2}} \\ &H(\xi) = [\tfrac{1}{2}(1+e^{-i\xi})]^2 \tfrac{1}{2}[1 + \sqrt{3} + (1-\sqrt{3})e^{-i\xi}] \end{aligned}$$

In the first example the function $\widehat{\phi}(\xi)$ can be computed easily. We have

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} \tfrac{1}{2}(1 + \exp(-i2^{-j}\xi)) = \frac{1 - e^{-i\xi}}{i\xi}.$$

Thus the farther wavelet $\phi(x)$ and the associated mother wavelet $\psi(x)$ are the Haar wavelets:

$$\phi(x) = \begin{cases} 1 & \text{as } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \qquad \psi(x) = \begin{cases} 1 & \text{as } 0 < x < \frac{1}{2} \\ -1 & \text{as } \frac{1}{2} < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

The first example is special with respect of symmetry because:

THEOREM 2. (Daubechies) *The Haar basis (associated to the above example (1)) is the only orthonormal basis of compactly supported wavelets for which the associated function* ϕ *has a symmetry axis.*

In [1] a constructive proof of the following theorem is given.

THEOREM 3. *Any trigonometric polynomial $H(\xi)$, which fulfils the conditions of Theorem 1, is of the form*

$$H(\xi) = [\tfrac{1}{2}(1+e^{-i\xi})]^N Q(e^{-i\xi}),$$

where $N \in \mathbb{N}$, $N \geq 1$, and where Q is a polynomial such that

$$|Q(e^{-i\xi})|^2 = \sum_{j=0}^{N-1} \binom{N-1+j}{j} \sin^{2j} \tfrac{1}{2}\xi + [\sin^{2N} \tfrac{1}{2}\xi] R(\tfrac{1}{2}\cos\xi),$$

where R is an odd polynomial, with some extra restrictions.

For obtaining the Daubechies' wavelets, we choose $R = 0$. Thus for fixed N the Daubechies' wavelets correspond to the trigonometric polynomials $H(\xi)$ of minimal degree. With some extra analysis it follows that these special trigonometric polynomials have a Q-term of the form

$$Q_N(e^{-i\xi}) = \sum_{n=0}^{N-1} q_n e^{-in\xi} \qquad \text{with} \qquad q_0 \neq 0. \tag{2.2}$$

EXAMPLES

(3)
$$Q_2(\xi) = \frac{1}{2}[1+\sqrt{3}+(1-\sqrt{3})e^{-i\xi}]$$

(4)
$$Q_3(\xi) = \tfrac{1}{4}\Big[1 + \sqrt{10} + \sqrt{5+2\sqrt{10}} + 2(1-\sqrt{10})e^{-i\xi} + \Big(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}\Big)e^{-2i\xi}\Big]$$

Let us denote the corresponding ϕ, ψ functions by ϕ_N, ψ_N. The theory of section 1 gives

$$\text{supp}(\phi_N) = [0, 2N-1], \qquad \text{supp}(\psi_N) = [-N+1, N].$$

For $N = 2$ and $N = 3$ the associated h_n can be calculated exactly, and for $N \leq 10$ Daubechies gives the numerical values of the h_n in [1].

When N increases with 1, the number of non-zero h_n increases with 2. And when N increases the smoothness of ϕ_N and ψ_N increases:

THEOREM 4. (Daubechies) *There exists $\lambda > 0$ such that, for all $N \in \mathbb{N}$, $N \geq 2$,*

$$\phi_N, \psi_N \in C^{\lambda N}$$

Daubechies gives some values of λ, for which Theorem 4 holds. But a larger value for which Theorem 4 holds, is due to MEYER [3], it is

$$\lambda \sim \log(\frac{4}{\pi}) / \log 2 \sim 0.3485.$$

3. Computing with the compactly supported Daubechies' wavelets

In this section we show how a Discrete Wavelet Transform of a function can be obtained. We will use the Daubechies' wavelet ϕ_N of the previous section, and, again, the associated functions are $\phi_{jk}(x) = 2^{-j/2}\phi_N(2^{-j}x - k)$. A function will be represented by a finite signal $\bar{a} = (a_0, \cdots, a_{M-1})^T$. As an intermediate expression we introduce $A = \sum_{j=0}^{M-1} a_j \phi_{Kj}$, with $M = 2^K$. Then the Discrete Wavelet Transform of $\bar{a}$ appears to be constituted by $2^K - 2$ coefficients of the expansion of A in the orthonormal ψ_{nm} basis.

Now we choose N fixed, and with the filter-coefficients h_n of ϕ_N we define filters $\mathbf{H}_N,\ \mathbf{G}_N : l^2 \longrightarrow l^2$

$$(\mathbf{H}_N \bar{a})_k = \sum_{l=-\infty}^{\infty} h_{l-2k} a_l, \tag{3.1}$$

$$(\mathbf{G}_N \bar{a})_k = \sum_{l=-\infty}^{\infty} g_{l-2k} a_l, \tag{3.2}$$

with again $g_n = (-1)^n h_{1-n}$.

Now we let these filters work on the finite signal $\bar{a} = (a_0, \cdots, a_{M-1})^T$, with M even. In matrix-form $\mathbf{H}_N$ is

$$\mathbf{H}_N = \begin{bmatrix} h_0 & h_1 & h_2 & & \cdots & h_{2N-1} & & & & \\ & & h_0 & h_1 & \cdots & & h_{2N-1} & & & \\ & & & & \ddots & & & & & \\ & & & & & & & & h_0 & h_1 \end{bmatrix} \tag{3.3}$$

The matrix-form of $\mathbf{G}_N$ is the same, with h_n replaced by g_n. Filtering with these $\frac{1}{2}M \times M$-matrices will cause edge effects for $N > 1$. For eliminating these edge effects, we make these matrices periodic in the following way

$$\mathcal{H}_N = \begin{bmatrix} h_0 & & \cdots & & h_{2N-1} & & & & \\ & & & & \ddots & & & & \\ & & & & & h_0 & & \cdots & h_{2N-1} \\ h_{2N-2} & h_{2N-1} & & & & & h_0 & \cdots & h_{2N-3} \\ & & \ddots & & & & & \ddots & \\ h_2 & & \cdots & h_{2N-1} & & & & h_0 & h_1 \end{bmatrix} \tag{3.4}$$

with $\mathcal{G}_N$ the same as $\mathcal{H}_N$ with h_n replaced by g_n. This is the same as leaving the filters $\mathbf{H}_N$ and $\mathbf{G}_N$ unchanged and make the signal $\bar{a}$ periodic to a l^∞-vector.

The total filtering is now

$$\begin{bmatrix} \mathcal{H}_N \\ \\ \mathcal{G}_N \end{bmatrix}_M \begin{pmatrix} a_0 \\ \vdots \\ a_{M-1} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_N \begin{pmatrix} a_0 \\ \vdots \\ a_{M-1} \end{pmatrix} \\ \mathcal{G}_N \begin{pmatrix} a_0 \\ \vdots \\ a_{M-1} \end{pmatrix} \end{pmatrix} : \mathbb{R}^M \to \mathbb{R}^M \tag{3.5}$$

The matrix $\begin{bmatrix} \mathcal{H}_N \\ \mathcal{G}_N \end{bmatrix}_M$ is orthonormal, i.e.

$$\begin{bmatrix} \mathcal{H}_N \\ \mathcal{G}_N \end{bmatrix}_M [\mathcal{H}_N^T \, \mathcal{G}_N^T]_M = \begin{bmatrix} \mathcal{H}_N\mathcal{H}_N^T & \mathcal{H}_N\mathcal{G}_N^T \\ \mathcal{G}_N\mathcal{H}_N^T & \mathcal{G}_N\mathcal{G}_N^T \end{bmatrix}_M = Id_M. \tag{3.6}$$

This follows directly from the equations

$$\sum_{n=-\infty}^{\infty} h_{n-2k}h_{n-2l} = \delta_{kl}, \tag{3.7}$$

$$\sum_{n=-\infty}^{\infty} h_{n-2k}g_{n-2l} = 0, \tag{3.8}$$

where (3.7) is condition (ii) of Theorem 1, and (3.8) follows from

$$\sum_{n=-\infty}^{\infty} h_{n-2k}g_{n-2l} = \sum_{n=-\infty}^{\infty} < \phi_{1k}, \phi_{0n} >< \phi_{0n}, \psi_{1l} >=< \phi_{1k}, \psi_{1l} >= 0.$$

We define

$$\phi_{mn}(x) = 2^{-\frac{m}{2}}\phi(2^{-m}x - n), \qquad \psi_{mn}(x) = 2^{-\frac{m}{2}}\psi(2^{-m}x - n), \tag{3.9}$$

and we choose $M = 2^K$, $K \in \mathbb{N}^*$. Let $A \in V_K$ such that

$$a_k = a_{k0} =< A, \phi_{Kk} > . \tag{3.10}$$

Then we define

$$a_{kj} =< A, \phi_{K-j,k} >, \qquad d_{kj} =< A, \psi_{K-j,k} > . \tag{3.11}$$

The Discrete Wavelet Transformation becomes (for $K = 4$)

$$\begin{pmatrix} a_{0,0} \\ a_{1,0} \\ a_{2,0} \\ a_{3,0} \\ a_{4,0} \\ a_{5,0} \\ a_{6,0} \\ a_{7,0} \\ a_{8,0} \\ a_{9,0} \\ a_{10,0} \\ a_{11,0} \\ a_{12,0} \\ a_{13,0} \\ a_{14,0} \\ a_{15,0} \end{pmatrix} \underset{\longrightarrow}{\begin{bmatrix} \mathcal{H}_N \\ \mathcal{G}_N \end{bmatrix}_{16}} \begin{pmatrix} a_{0,1} \\ a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ a_{4,1} \\ a_{5,1} \\ a_{6,1} \\ \underline{a_{7,1}} \\ d_{0,1} \\ d_{1,1} \\ d_{2,1} \\ d_{3,1} \\ d_{4,1} \\ d_{5,1} \\ d_{6,1} \\ d_{7,1} \end{pmatrix} \underset{\longrightarrow}{\begin{bmatrix} \mathcal{H}_N \\ \mathcal{G}_N \end{bmatrix}_{8}} \begin{pmatrix} a_{0,2} \\ a_{1,2} \\ a_{2,2} \\ \underline{a_{3,2}} \\ d_{0,2} \\ d_{1,2} \\ d_{2,2} \\ \underline{d_{3,2}} \\ d_{0,1} \\ d_{1,1} \\ d_{2,1} \\ d_{3,1} \\ d_{4,1} \\ d_{5,1} \\ d_{6,1} \\ d_{7,1} \end{pmatrix} \underset{\longrightarrow}{\begin{bmatrix} \mathcal{H}_N \\ \mathcal{G}_N \end{bmatrix}_{4}} \begin{pmatrix} a_{0,3} \\ \underline{a_{1,3}} \\ d_{0,3} \\ \underline{d_{1,3}} \\ d_{0,2} \\ d_{1,2} \\ d_{2,2} \\ \underline{d_{3,2}} \\ d_{0,1} \\ d_{1,1} \\ d_{2,1} \\ d_{3,1} \\ d_{4,1} \\ d_{5,1} \\ d_{6,1} \\ d_{7,1} \end{pmatrix}. \tag{3.12}$$

So, for general K, the Discrete Wavelet Transformation is built up by simple matrix-operations with orthonormal matrices. The special form of these matrices make these matrix-operations, and the Discrete Wavelet Transform for general K, easy to program. And the whole process is simple to invert. Notice that the transformation is a process that transforms a signal of length 2^K into a vector of length 2^K.

The d_{jk} are some of the coefficients of the expansion of A in the orthonormal ψ_{jk} basis. The remaining $a_{0,K-1}$ and $a_{1,K-1}$ are called the "mother-function coefficients".

4. What do Daubechies' wavelets look like, and how do they work on signals?

The illustrations in this section are made on a Macintosh-II using a Pascal program, which is based on the program given in PRESS [4]. In the illustrations,

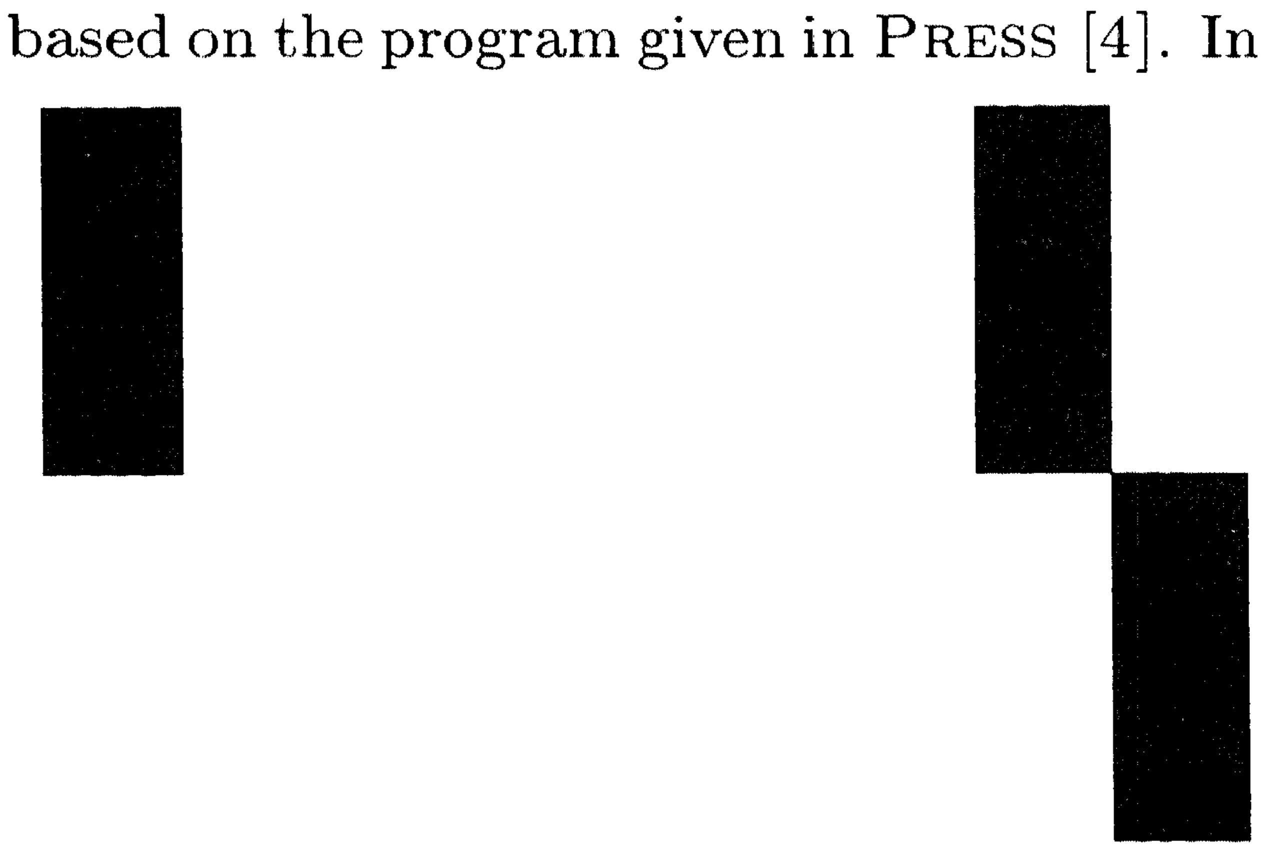

FIGURE 1. *Father (left) and mother wavelet for $N = 1$, the Haar functions.*

the length of the signals $M = 1024$. First we show some figures of the ϕ_N and ψ_N for some different values of N. Notice that a signal of ψ_N can be obtained by starting at the right hand side of scheme (3.12) with a vector $\delta_i = \{\delta_{ik}\}_{k=1}^{1024}$, where $i \in 2, \cdots, 1024$ is fixed. This follows directly from (3.11). The figures are shown in Figure 1, 2 and 3.

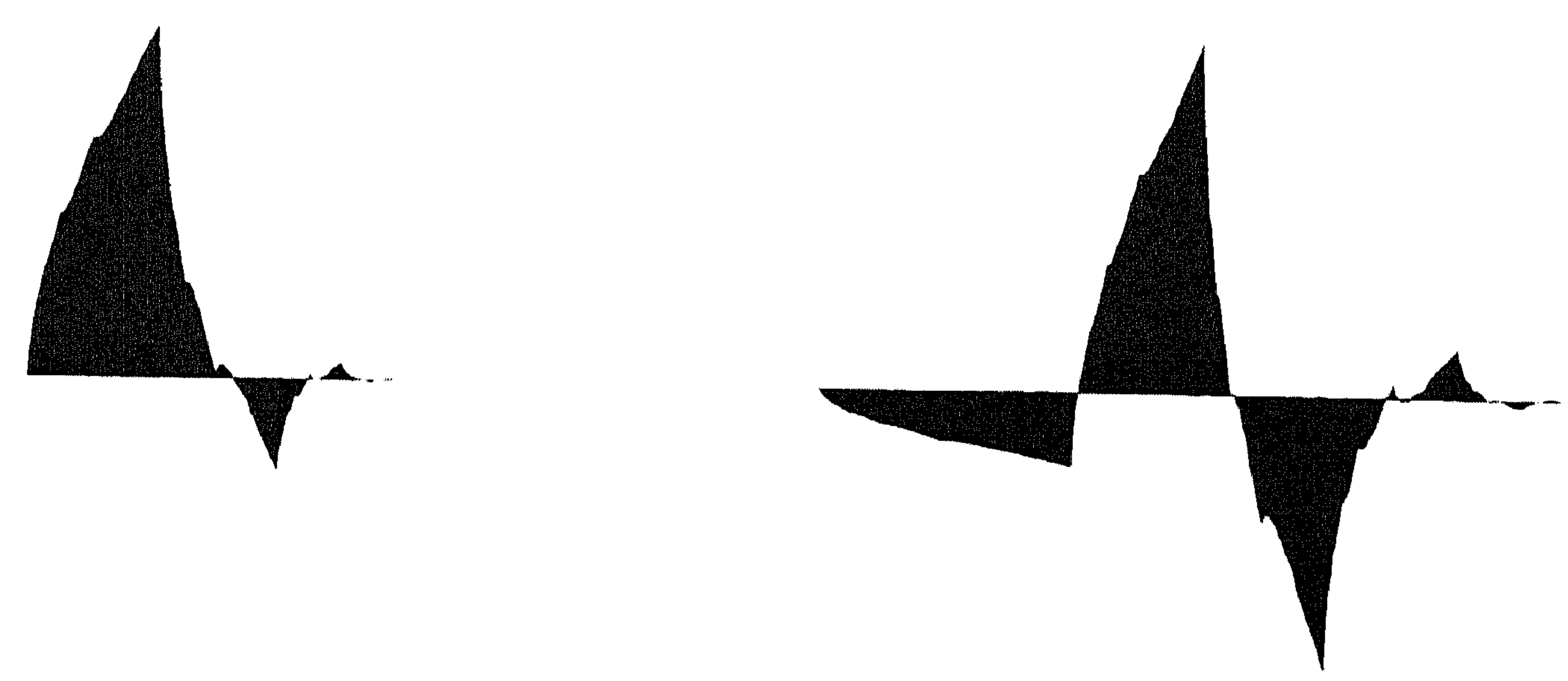

FIGURE 2. *Father (left) and mother wavelet for $N = 2$.*

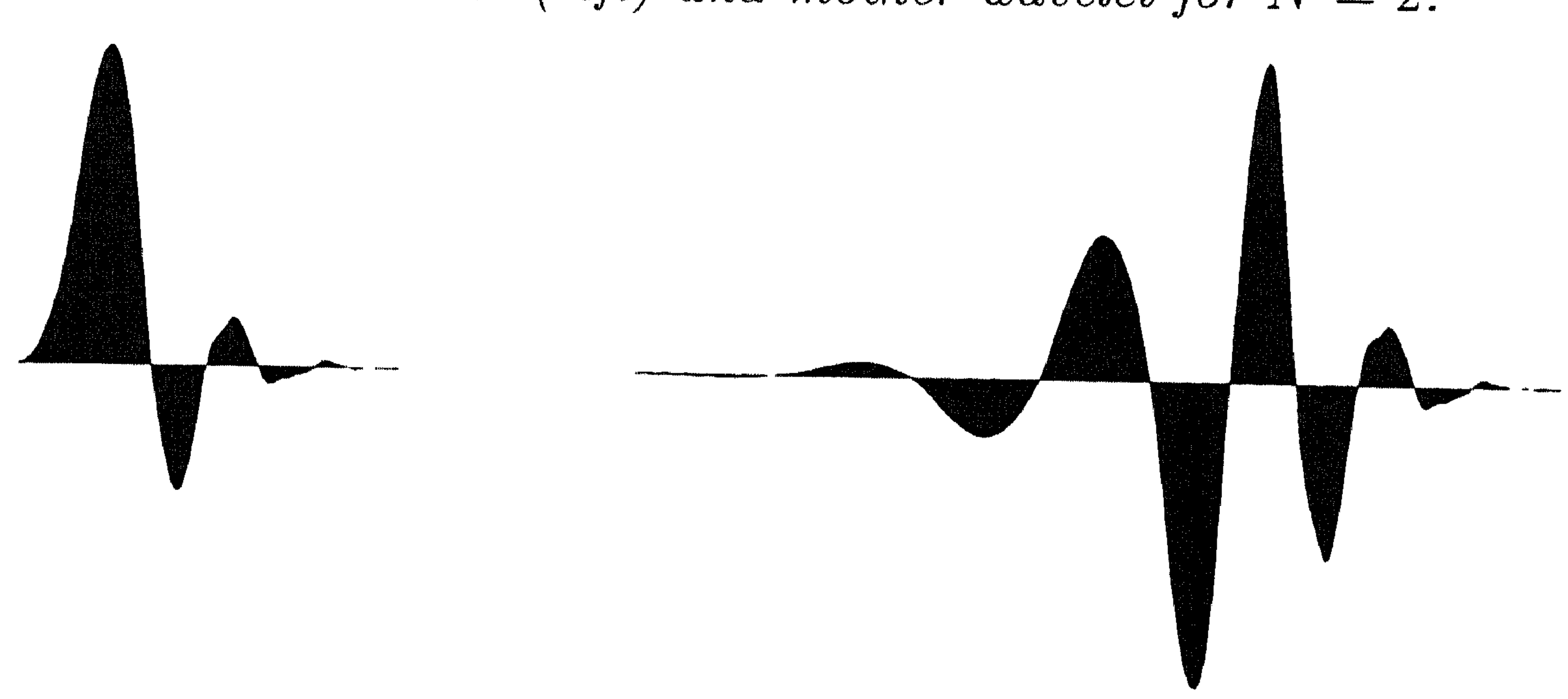

FIGURE 3. *Father (left) and mother wavelet for $N = 6$.*

Notice that, according to Theorem 2 and 4, the smoothness of the wavelets increases with N, and only for $N = 1$ the ϕ_N has a symmetry axis. The inverse discrete wavelet transform of $\delta_{10} + \delta_{58}$ is shown in Figure 4.

FIGURE 4. *The inverse discrete wavelet transform of $\delta_{10} + \delta_{58}$ for $N = 2$.*

Since 10 lies early in the hierarchical range 9-16, that wavelet lies on the left side of the picture. Since 58 lies in a later (smaller scale) hierarchy, it is a narrower wavelet.

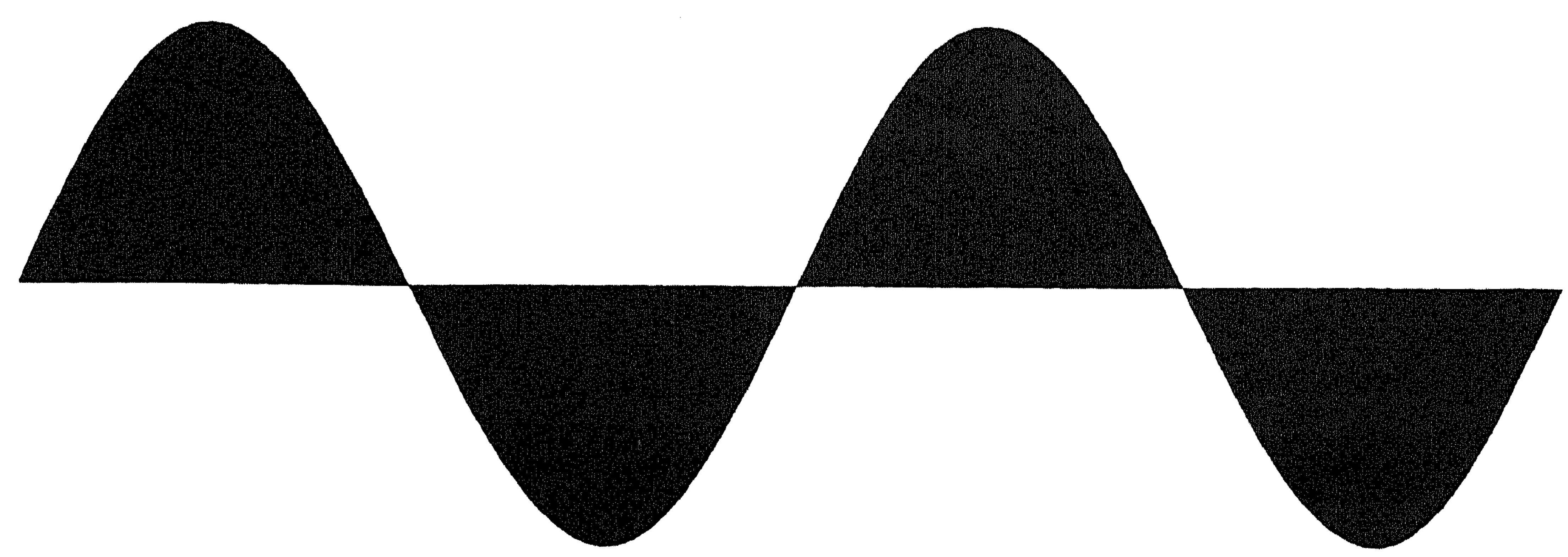

FIGURE 5. *The sinus-signal.*

The discrete wavelet transforms of the sinus-signal (see Figure 5) are calculated for $N = 1$ and $N = 10$. They are shown in Figures 6 and 7.

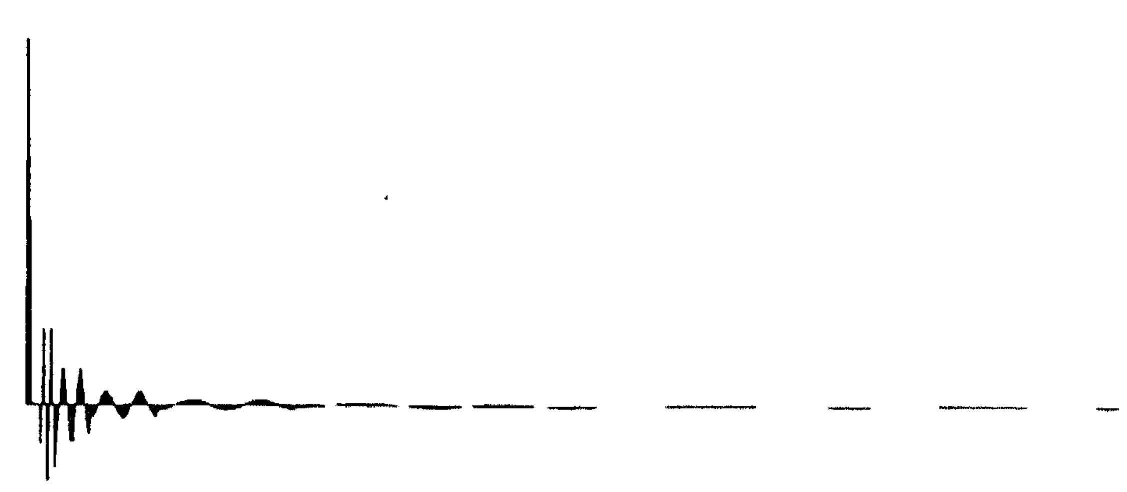

FIGURE 6. *Discrete wavelet transform of the sinus-signal for* $N = 1$.

The amplitudes of the transform for $N = 10$ are more concentrated at the left part of the signal then the amplitudes of the transform for $N = 1$. This can be explained by the smoothness of the underlying wavelets. The wavelets for $N = 10$ match better to the smooth signal. In other cases, when the original signal is more singular, the wavelets for $N = 1$ match better to that signal.

FIGURE 7. *Discrete wavelet transform of the sinus-signal for* $N = 10$.

Now we can truncate these wavelet transforms. For $N = 1$ there are 18 coefficients that have amplitudes larger then 0.05 times the maximum amplitude of the transform. We set the remaining amplitudes to zero. With this kind of data-reduction we have to record both the values and the positions of the non-zero coefficients. Thus in the case of $N = 1$ we reduce to a vector of length 36. And in the case of $N = 10$ we reduce to a vector of length 12. The following two pictures show the result of truncations of the original signal from the two inverse discrete wavelet transforms of the truncated vectors.

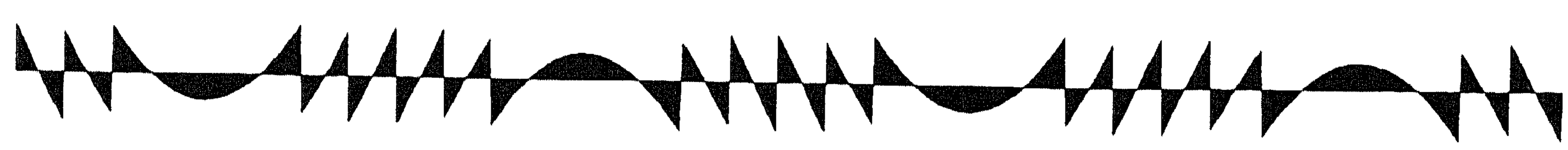

FIGURE 8. *Original signal minus approximation signal for $N = 1$.*

FIGURE 9. *Original signal minus approximation signal for $N = 10$.*

So, when we start with a smooth signal, the data-reduction with $N = 10$ is much better.

REMARK It is very important that vectors in wavelet space be truncated according to the amplitude of the components, not their position in the vector. Keeping the first 16 components of the vector would give an extremely poor approximation to the original signal.

REFERENCES

1. DAUBECHIES, I. (1988). *Orthonormal bases of compactly supported wavelets,* Comm. Pure Appl. Math., **41**, 909-996.
2. LEMARIÉ-RIEUSSET, P.G. and MALGOUYRES, G. (1991). *Support des fonctions de base dans une analyse multi-résolution,* C.R. Acad. Sci. Paris Sér. I Math., **313**, 377-380.
3. MEYER, Y. (1987). *Wavelets with compact support,* Zygmund Lectures, University of Chicago.
4. PRESS, W.H. (1991). *Wavelet transforms: a primer,* preprint